

# OLS with two fixed effects

Die Mathematiker sind eine Art Franzosen: redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes. – J.W.v. Goethe

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# Our corner of reality

Consider a model

$$y_{ijt} = X_{ijt}\beta + \alpha_i + \gamma_j + \epsilon$$

where  $y_{ijt}$  is log-wage of individual  $i$  in firm  $j$  at time  $t$ .  $X_{ijt}$  is a set of time-varying covariates for individual  $i$  and firm  $j$ ,  $\alpha_i$  is an individual fixed effect, and  $\gamma_j$  is a firm-fixed effect.

- ▶ Such models have been used to study correlations between individual effects and firm-effects (“High wage workers and high wage firms”). It picks up arbitrary unobserved heterogeneity in both firms and workers. Correlation between  $\alpha$ ,  $\gamma$  and  $X$  is allowed.
- ▶ One might imagine other fixed effects, such as students and schools, or scientists and journals, citizens and their home-location. Or even 3 or more fixed effects.
- ▶ In some cases we might not be interested in the  $\alpha$ ’s and  $\gamma$ ’s, we only need them as controls.
- ▶ These models have been difficult to estimate due to the high number of dummy-variables. (One for each firm, one for each individual)

## Single fixed effect - familiar within groups estimator

- ▶ When only considering one fixed effect, e.g. individual fixed effects, the “within-groups” estimator is frequently used.

$$Y = X\beta + D\alpha + \epsilon$$

- ▶ Subtract the group-means from  $Y$  and  $X$ , and find  $\hat{\beta}_{FE}$  from the system

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- ▶ “Time-demeaning”, “centering on the means”, “creating mean deviations”, “sweeping out the fixed effects”. Many names for this. I like centering and sweeping.
- ▶ If we need  $\hat{\alpha}$  too, i.e. not only to control for fixed effects, we may recover  $\hat{\alpha}$  by solving  $D'D\hat{\alpha} = D'(Y - X\hat{\beta}_{FE})$ . This is easy since  $D'D$  is diagonal.  $\hat{\alpha}$  turns out to be the group means of the residuals.

## Two (or more) fixed effects — ineffably tangled mess

- ▶ With two fixed effects (“worker”  $\alpha$  and “firm”  $\gamma$ )

$$Y = X\beta + D_1\alpha + D_2\gamma + \epsilon \quad (1)$$

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- ▶ Thus, various elaborate estimation schemes have been concocted during the last decade; including sweeping out *one* of the fixed effects, iterated estimations, linear approximations, conjugate gradients with ingenious preconditioners; with all sorts of mathematical formulæ with triple indices, double sums, and even *square* roots.



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- ▶ This is the wrong way to think about this problem, we’re not supposed to sweep out workers from their firms, but to sweep out workers *and* firms from eq. (1).

## Street sweepers — John von Neumann and Israel Halperin

- By doing Gaussian elimination on the normal equations, we see that we need to center on both means at once; like this  
 $\bar{Y} = PY; \quad \bar{X} = PX$  where  $P$  is the projection onto the orthogonal complement of the column space of  $D = [D_1 \ D_2]$ ; (i.e.  $R(D)^\perp$ ).

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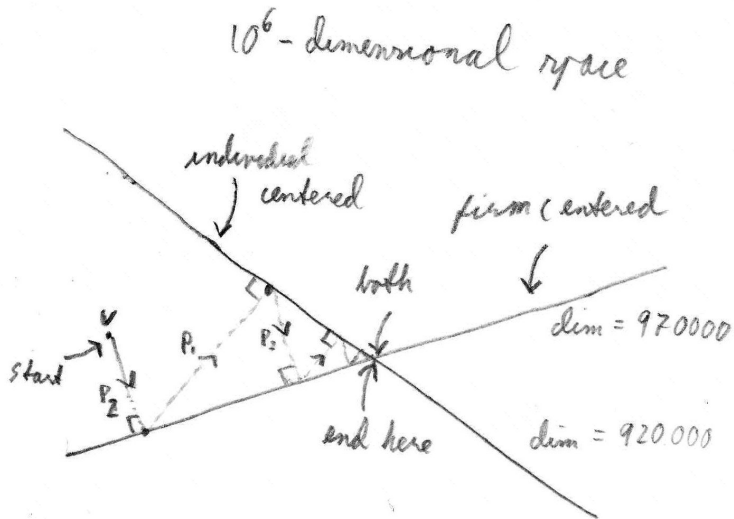
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- ▶ Thus, we may sweep out both (or more) fixed effects by centering  $Y$  and  $X$  on the individual means, then on the firm means, then on the individual means again, then the firm means, individual means, firm means, i.m., f.m., ..., until ... they're gone.

## A graphical rendition of the process





## So ... What about the fixed effects?

- ▶ Once we have acquired  $\hat{\beta}$ , we compute the residuals  $R = Y - X\hat{\beta}$ , go back to the normal equations and note that  $D'D \begin{bmatrix} \hat{\alpha} \\ \hat{\gamma} \end{bmatrix} = D'R$  where  $D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$ . (Just like the within-groups estimator).

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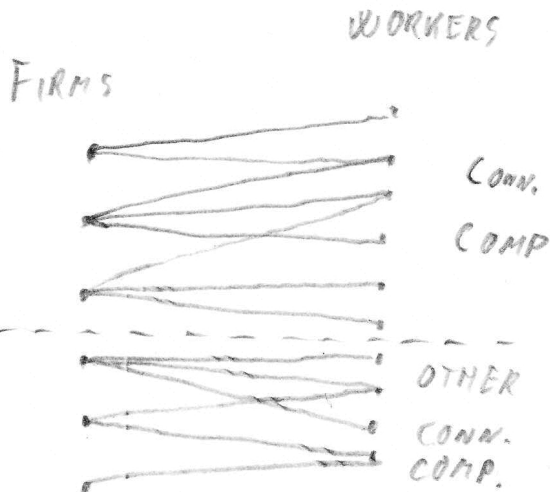
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- ▶ Consider a case where we have 15000 firms with 300000 employees moving between them. Then we have another set of 9000 firms, with a set of 120000 employees. But no employee in one group ever moves into the other group. In each group we may add a constant  $c$  to every individual effect, and subtract the same constant from every firm effect, and we will not be able to tell:

$$y_{ijt} = x_{ijt}\beta + (\alpha_i + c) + (\gamma_j - c) = x_{ijt}\beta + \alpha_i + \gamma_j$$

We may have a different constant  $c$  in each group, thus estimates are not comparable across groups.

# Connection components



# That's all there is to it

- ▶ Divide the dataset into (graph-theoretic) connection components (of firms and individuals, with no relation to the other components).
- ▶ In each component we must pick a reference (either a firm or an employee). (This is due to a somewhat lengthy argument by Abowd et al, but is also a direct consequence of a theorem about the signless Laplacian spectrum of bipartite graphs). Then we may solve for the fixed effects separately in each component, like

$$B'B \begin{bmatrix} \hat{\alpha} \\ \hat{\gamma} \end{bmatrix} = B'R \text{ (these are "sparse" systems, i.e. mostly zeroes).}$$

All the systems are conditional on the jointly estimated  $\hat{\beta}$  (via the residuals  $R = Y - X\hat{\beta}$ ).

- ▶ The fixed effects are identified up to adding a constant  $c$  to the individual effects and subtracting the same  $c$  from the firm effects. Thus, differences (and variances) between individual effects within a component are identified, and so are differences between firm effects.

## Practical estimation

We have installed software (under the name “LFE” - linear fixed effects) doing the above on “leif”, our compute cluster. Both centering and solving the component systems are *embarrassingly parallel* tasks, thus we do this in parallel over 8 cpus. A typical specification file looks like this

```
file middata.csv           # name of data file
vars x x2 year id firm y ife ffe yfe # layout of data-file
model y ~ x + x2 + year     # R-style model-spec
dummy year                 # tell'em it's categories
complim 10                 # ditch small components
fe firm id                 # fixed effects
```

A package “lfe” will shortly be uploaded to “CRAN”, the R package repository for public download. (Our installed “LFE” is just a wrapper around this package.)

# Elsewhere

This way of finding the intersection of projections is known as *MAP - Method of Alternating Projections* and has been in use in image processing, not in Photoshop (I think), but in computed tomography (CT) for a while, where it is known as *ART - Algebraic Reconstruction Technique*. In numerical linear algebra it is known as *The Kaczmarz Method*.



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